RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. SECOND SEMESTER EXAMINATION, MAY 2012

FIRST YEAR

MATHEMATICS (Honours)

Date : 21/05/2012

Time : 11 am – 3 pm

Paper : II

Full Marks : 100

[Use Separate Answer Books for each group]

Group - A

Answer **any five** from question no. 1-8 and **any five** from question no. 9-16.

1. If x, y, z are positive real numbers and $x+y+z=1$, prove that $8xyz \le (1-x)(1-y)(1-z) \le \frac{8}{27}$. [5]

2. a) If $x_i > a > 0$ for i = 1, 2, ..., n and $(x_1-a)(x_2-a)...(x_n-a) = K^n (K>0)$, prove that $x_1x_2...x_n \ge (a+K)^n$. [3]

   b) Find the maximum value of $(x+2)^5(7-x)^4$ when $-2<x<7$. [2]

3. State De Moiver's theorem for rational index. Find all the distinct values of $(1+i\sqrt{3})^{3/4}$ and show that the product of all the distinct values of $(1+i\sqrt{3})^{3/4}$ is 8. [2+3]

4. a) If x be a negative real number then prove that $i\log\frac{x+i}{x-i} = \pi + 2\tan^{-1}x$. [2]

   b) Find the general values and principal value of $(1+i)^{1+i}$. [3]

5. Find the equation whose roots are squares of the roots of $x^4 - x^3 + 2x^2 - x + 1 = 0$ and use Descartes' rule of signs to show that the given equation has no real root. [5]

6. Solve the equation : $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$. [5]

7. Find the special roots of $x^{15} - 1 = 0$. Deduce that $2\cos\frac{2\pi}{15}, 2\cos\frac{4\pi}{15}, 2\cos\frac{8\pi}{15}, 2\cos\frac{16\pi}{15}$ are the roots of $x^4 - x^3 - 4x^2 + 4x + 1 = 0$. [2+3]

8. Solve by Ferrari's method : $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$. [5]

9. a) If A and B are respectively closed and compact subsets of the set of real numbers $\mathbb{R}$,then show that $A \cap B$ is compact. [2]

   b) Let $H = [0,\infty)$ and $I_n = (-1,n), n \in \mathbb{N}$. Show that $G = \{I_n, n \in \mathbb{N}\}$ is an open cover of H but no finite subcollection of G can cover H. [3]

10. Let $\sum_{n=1}^{\infty} u_n$ be a series of positive real numbers and let $w_n = \frac{u_n}{u_{n+1}}\sqrt{n} - \sqrt{n+1}$. If $\underline{\lim}\ w_n = K > 0$ then prove that $\sum_{n=1}^{\infty} u_n$ is convergent and if $\overline{\lim}\ w_n = K < 0$ then prove that $\sum_{n=1}^{\infty} u_n$ is divergent. [5]

11. a) Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^n}{(n+1)^{n+1}}$ is convergent. [2]

b) Prove that the series $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+....$ converges to $\frac{3}{2}\log 2$. (It may be assumed that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}....$ converges to log 2). [3]

12. a) Prove that the series $\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)^p+....$ is convergent for $p>2$ and divergent for $p\le 2$. [3]

b) Let $\sum_{K=1}^{\infty} a_K$ be a convergent series of positive real numbers and $\{a_{n_K}\}$ be a subsequence of $\{a_K\}$.

Prove that $\sum_{K=1}^{\infty} a_{n_K}$ converges. [2]

13. Let $A=(0,\infty)$ and $f: A\to\mathbb{R}$ be defined as : f(x) = 0 when x is irrational

$= n$ when $x=\frac{m}{n}$ with m, n $\in\mathbb{N}$ and g.c.d (m, n) = 1.

Prove that f is discontinuous on A. [5]

14. a) When a function $f: I\to\mathbb{R}$ is said to be uniformly continuous on $I(\subset\mathbb{R})$? Give an example with proper justification of a function continuous over the open interval (0,1) but not uniformly continuous thereon. [3]

b) Let $f:[0,1]\to\mathbb{R}$ be continuous on [0,1] and f assumes only rational values. If $f\left(\frac{1}{2}\right)=\frac{1}{2}$, prove that $f(x)=\frac{1}{2}$ for all $x\in[0,1]$. [2]

15. Let $f:[a,b]\to\mathbb{R}$ be differentiable on [a, b]. Let $f'(a)<f'(b)$ and K be any real number between $f'(a)$ and $f'(b)$. Prove that there exists $c\in(a,b)$ such $f'(c)=K$. [5]

16. a) Answer either (i) or (ii) : [2]

i) Find a and b such that $\lim_{x\to 0}\frac{ae^x+be^{-x}+2\sin x}{\sin x+x\cos x}=2$.

ii) Prove that $\lim_{x\to 0+} x^x=1$.

b) Find the points of local maximum and local minimum of the function f given by $f(x)=\sin^{-1}(2x\sqrt{1-x^2}), x\in(-1,1)$. [3]

Group - B

Answer **any two** from question no. 17-19 and **any three** from question no. 20-24.

17. a) If A be a symmetric matrix of order m and P be an m×n matrix, prove that P^tAP is a symmetric matrix. [3]

b) If the planes $x=cy+bz$, $y=az+cx$, $z=bx+ay$ have a common point other than the origin, prove that $a^2+b^2+c^2+2abc=1$. [3]

c) If $(I+A)^{-1}(I-A)$ is a real orthogonal matrix, prove that the matrix A is skew-symmetric. [4]

18. a) Find the rank of the matrix A by reducing it to Normal form : $A=\begin{bmatrix}8 & 1 & 3 & 6\\ 0 & 3 & 2 & 2\\ -8 & -1 & -3 & 4\end{bmatrix}$ [4]

b) Show that the linear sum of two subspaces of a vector space V over a field F is again a subspace of V. [3]

c) If a set of vectors $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ in a vector space V over a field F be linearly dependent, then prove that at least one of the vectors of the set can be expressed as a linear combination of the remaining others. [3]

19. a) Prove that the set $\{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis of $V_3(R)$. Find the coordinates of the vector (a,b,c) with respect to the above basis. [3+2]

b) Define subspace of a vector space. Show that the set $S = \{(x,y,z,w) \in R^4 / x+y-2z+w=0, 2x+y-3w=0\}$ is a subspace of the real vector space R^4. Find also the Dimension of the Subspace. [1+2+2]

20. a) A transport company has offices in five localities A, B, C, D and E. Some day the officers located at A and B had 8 and 10 spare trucks, whereas the officers at C, D, E required 6, 8, 4 trucks respectively. The distance in kilometre between the five localities given below :

To	C	D	E
From A	2	5	3
B	4	2	7

How should the trucks from A and B be sent to C, D, E so that the total distance covered by the trucks is minimum. Formulate the problem as a linear programming problem. [4]

b) How many basic solutions are there in the following set of equations?

$$2x_1 - 5x_2 + x_3 + 3x_4 = 4$$

$$3x_1 - 10x_2 + 2x_3 + 6x_4 = 14$$

Justify your answer. Find all basic solutions and basic feasible solutions. [4]

c) Examine if S is a convex polyhedron where $S = \{(x,y) \in E^2; x+3y \le 3, 2x-y=4, x \ge 0, y \ge 0\}$. [2]

21. a) Solve the following assignment problem. [4]

Project	Location				
	I	II	III	IV	V
A	15	21	6	4	9
B	3	40	21	10	7
C	9	6	5	8	10
D	14	8	6	9	3
E	21	16	18	7	4

b) Solve the following transportation problem : [6]

					a_i
	19	20	50	10	7
	70	30	40	60	20
	40	8	70	20	18
b_j	5	8	7	14	

22. a) Solve by simplex method : [5]

Minimize $z = x_1 - 3x_2 + 2x_3$ subject to the constraints

$$3x_1 - x_2 + 2x_3 \le 7$$

$$-2x_1 + 4x_2 \le 12$$

$$-4x_1 + 3x_2 + 8x_3 \le 10$$

$$x_1, x_2, x_3 \ge 0$$

b) Solve by two phase Method [5]

Minimize $z = 3x_1 + 5x_2$ subject to the constraints

$$x_1 + 2x_2 \geq 8$$

$$3x_1 + 2x_2 \geq 12$$

$$5x_1 + 6x_2 \leq 60$$

$$x_1, x_2 \geq 0$$

23. a) Write the dual of the problem [4]

Minimize $Z = x_1 - 3x_2 + 2x_3$ subject to the constraints

$$x_1 - x_2 - x_3 + x_4 = 8$$

$$3x_1 + x_3 - 2x_4 \geq 9$$

$x_1, x_2, x_3 \geq 0, x_4$ unrestricted in sign.

b) Solve the following Travelling and Salesman Problem [6]

	A	B	C	D	E
A	–	1	4	7	1
B	3	–	2	7	2
C	8	6	–	4	6
D	9	3	5	–	7
E	1	2	2	7	–

24. a) Show that all B.F.S of the set of equations Ax = b, $x \geq 0$ are extreme points of the convex set of feasible solutions of the equations. [5]

b) Given that $x_1 = 1, x_2 = 3, x_3 = 2$ is F.S. of the equation [3]

$$2x_1 + 4x_2 - 2x_3 = 10$$

$$10x_1 + 3x_2 + 7x_3 = 33$$

Reduce the above F.S. to a B.F.S by reduction theorem.

c) State fundamental theorem of linear programming problem. [2]